CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS

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Introduction. Let $\{X_k : k \geqq 1\}$ denote a sequence of random variables, $\{a_n : n \geqq 1\}$ a sequence of real numbers, $\{b_n : n \geqq 1\}$ a nondecreasing sequence of positive real numbers and let $S_n = \sum_{k=1}^{n} X_k$. Many of the limit theorems of probability theory may then be formulated as theorems concerning the convergence of either the sequence $\{P(|(S_n - a_n)/b_n| > \varepsilon) : n \geqq 1\}$ or $\{P(\sup_{k \geqq n} |(S_k - a_k)/b_k| > \varepsilon) : n \geqq 1\}$, for $\varepsilon > 0$, to an appropriate limiting value. It is the purpose of this paper to study the rates of convergence of such sequences. The results of this paper will include those previously announced in [1].

In the first part of the paper attention is restricted to sequences of independent and identically distributed random variables. In analogy with the Law of Large Numbers the normalizing constants b_n are chosen to be n^α, $\alpha > 1/2$, and the centering constants $a_n = ES_n$, provided the expectation exists and is finite. Necessary and sufficient conditions are found, in terms of the order of magnitude of $P(|X_k| > n)$, for the sequences $\{P(|(S_n - ES_n)/n^\alpha| > \varepsilon) : n \geqq 1\}$ and $\{P(\sup_{k \geqq n} |(S_k - ES_k)/k^\alpha| > \varepsilon) : n \geqq 1\}$ to converge to zero at specified rates. These results extend and complete previous work on this problem.

The next results, again for independent and identically distributed random variables, consider the problem of convergence rates when the b_n's are a sequence in the upper class of the S_n's and again $a_n = ES_n$. Next the independence conditions on the sequence $\{X_k : k \geqq 1\}$ are relaxed and it is assumed only that the random variables form a stationary sequence. It is shown here that no conditions on the size of the variables, i.e. conditions on the magnitude of $P(|X_k| > n)$, can insure a prescribed rate of convergence of $P(|(S_n - ES_n)/n| > \varepsilon)$ to zero when the X_k's form an ergodic stationary sequence. However, in the converse direction, a prescribed rate of convergence to zero of the above probabilities does imply conditions on the magnitude of $P(|X_k| > n)$.

Finally in the last section of the paper some propositions and examples are presented for the case of independent but not necessarily identically distributed random variables.

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Independent, identically distributed random variables. In this section only sequences $\{X_k: k \geqq 1\}$ of independent and identically distributed random variables will be considered. Given a sequence $\{c_n: n \geqq 1\}$ of bounded, non-negative numbers converging to zero, one method of measuring the rate of convergence is to determine which, if any, of the series $\sum_{n=1}^{\infty} n^r c_n$ converge where $r \geqq -1$. This is the idea behind the first three theorems of this section.

THEOREM 1. *Let* $0 < t < 1$. *The following two statements are equivalent:*
(a) $E|X_k|^t < \infty$;
(b) $\sum_{n=1}^{\infty} n^{-1} P\{|S_n| > n^{1/t}\varepsilon\} < \infty$ *for all* $\varepsilon > 0$.
Let $1 \leqq t < 2$. *The following two statements are equivalent:*
(c) $E|X_k|^t < \infty$ *and* $EX_k = \mu$;
(d) $\sum_{n=1}^{\infty} n^{-1} P\{|S_n - n\mu| > n^{1/t}\varepsilon\} < \infty$ *for all* $\varepsilon > 0$.

Proof. First it will be shown that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d). Assume, with no loss of generality, that $\varepsilon = 1$ and, if EX_k exists, that $E(X_k) = 0$. For $n = 1, 2, \cdots$ and $k = 1, 2, \cdots, n$ define

$$X_{kn} = \begin{cases} X_k & \text{if } |X_k| < n^{1/t}, \\ 0 & \text{otherwise} \end{cases}$$

and observe that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1} P\{|S_n| > n^{1/t}\} \\ (1) \qquad &\leqq \sum_{n=1}^{\infty} n^{-1} P\{|X_k| \geqq n^{1/t} \text{ for some } k \leqq n\} \\ &\quad + \sum_{n=1}^{\infty} n^{-1} P\left\{ \left|\sum_{k=1}^{n} (X_{kn} - EX_{kn})\right| > n^{1/t}(1 - n^{1-1/t}|EX_{kn}|)\right\}. \end{aligned}$$

The first series on the right-hand side of (1) converges since

$$\sum_{n=1}^{\infty} n^{-1} P\left\{ |X_k| \geqq n^{1/t} \text{ for some } k \leqq n \right\} \leqq \sum_{n=1}^{\infty} P\{|X_k| \geqq n^{1/t}\}$$

and the finiteness of the last series is equivalent to $E|X_k|^t < \infty$.

For $t \geqq 1$ it follows from integration by parts and the fact that $EX_k = 0$ that $n^{1-1/t}|EX_{kn}| \to 0$ as $n \to \infty$; for $0 < t < 1$

$$n^{1-1/t}|EX_{kn}| \leqq n^{1-1/t} \int_{|x| < n^{1/t}} |x|^t \{|x|/n^{1/t}\}^{1-t} dF(x) \to 0.$$

Thus to show that the second series on the right side of (1) is finite it is enough to show that $\sum_{n=1}^{\infty} n^{-1} P\{|\sum_{k=1}^{n}(X_{kn} - EX_{kn})| > cn^{1/t}\} < \infty$ for some c, $0 < c < 1$. This is done as follows:

$$\sum_{n=1}^{\infty} n^{-1} P\left\{\left|\sum_{k=1}^{n}(X_{kn}-EX_{kn})\right|>cn^{1/t}\right\} \leqq c\sum_{n=1}^{\infty} n^{-1-2/t}E\left\{\sum_{k=1}^{n}(X_{kn}-EX_{kn})\right\}^2$$

$$\leqq c\sum_{n=1}^{\infty} n^{-1-2/t}\sum_{k=1}^{n} EX_{kn}^2 = c\sum_{n=1}^{\infty} n^{-2/t}\int_{|x|<n^{1/t}} x^2F(dx)$$

$$(2)\qquad \leqq c\sum_{n=1}^{\infty} n^{-2/t}\sum_{k=1}^{n} k^{2/t}P\{k-1\leqq |X_1|^t<k\}$$

$$= c\sum_{k=1}^{\infty} k^{2/t}P\{k-1\leqq |X_1|^t<k\}\cdot\sum_{n=k}^{\infty} n^{-2/t}$$

$$\leqq c\sum_{k=1}^{\infty} kP\{k-1\leqq |X_1|^t<k\}<\infty.$$

Note that in (2) and throughout this paper c denotes all constants and thus even in a single inequality c can denote two different values.

To prove the converse assertions we may assume $EX_k=0$. The proof proceeds by showing first that $S_n^s/n^{1/t}\to 0$ in probability, where X^s will always denote the symmetrized random variable X. Assume that $S_n^s/n^{1/t}$ does not converge in probability to zero. Then there exists $\varepsilon>0$ such that either $P\{S_{n_i}^s/n_i^{1/t}>\varepsilon\}>\varepsilon$ or $P\{S_{n_i}^s/n_i^{1/t}<-\varepsilon\}>\varepsilon$ for infinitely many i. For argument's sake assume $P\{S_{n_i}^s/n_i^{1/t}>\varepsilon\}>\varepsilon$ for infinitely many i. With no loss of generality choose $n_{i+1}>2n_i$. Now for each j such that $n_i<j\leqq 2n_i$ it follows by symmetry that $P\{\sum_{k=n_i+1}^{j} X_k^s\geqq 0\}\geqq 1/2$ and thus $P\{\sum_{k=1}^{j} X_k^s\geqq j^{1/t}\varepsilon/2^{1/t}\}\geqq \varepsilon/2$ for $n_i\leqq j\leqq 2n_i$. Therefore

$$(3)\qquad \sum_{n=1}^{\infty} n^{-1}P\{S_n^s/n^{1/t}\geqq \varepsilon/2^{1/t}\}\geqq \sum_{i=1}^{\infty}\sum_{n=n_i}^{2n_i} n^{-1}P\{S_n^s/n^{1/t}\geqq \varepsilon/2^{1/t}\} \geqq \frac{\varepsilon}{2}\sum_{i=1}^{\infty}\sum_{n=n_i}^{2n_i} n^{-1}=\infty.$$

However, (3) implies that $\sum_{n=1}^{\infty} n^{-1}P\{|S_n|/n^{1/t}\geqq \varepsilon/2^{1/t+1}\}=\infty$, a contradiction. Thus $S_n^s/n^{1/t}\to 0$ in probability and in addition we also conclude that $nP\{|X_k^s|>n^{1/t}\varepsilon\}\to 0$ as $n\to\infty$ for all $\varepsilon>0$. We may thus proceed exactly as in [3] to prove that $\sum_{n=1}^{\infty} P\{|X_1^s|>n^{1/t}\}<\infty$. Therefore $E|X_1^s|^t<\infty$ and consequently $E|X_k|^t<\infty$. Finally, if $t\geqq 1$, it follows from Marcinkiewicz's theorem [8, p. 242] that $(S_n-ES_n)/n^{1/t}\to 0$ almost surely and therefore $EX_k=0$. This completes the proof.

In the case $t=1$ the above theorem has been proved in [9] by entirely different methods. The methods here employed are much more elementary and have the further advantage that they can be applied to give unified and simplified proofs of all the results of this type.

THEOREM 2. *Let* $0<t<1$. *The following three statements are equivalent*:

(a) $E|X_k|^t \lg^+ |X_k| < \infty$;
(b) $\sum_{n=1}^{\infty} n^{-1} \lg n P\{|S_n| > n^{1/t}\varepsilon\} < \infty$ *for all* $\varepsilon > 0$;
(c) $\sum_{n=1}^{\infty} n^{-1} P\{\sup_{k \geq n} |S_k/k^{1/t}| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$.

Let $1 \leqq t < 2$. *The following three statements are equivalent*:
(d) $E|X_k|^t \lg^+ |X_k| < \infty$ *and* $EX_k = \mu$;
(e) $\sum_{n=1}^{\infty} n^{-1} \lg n P\{|S_n - n\mu| > n^{1/t}\varepsilon\} < \infty$ *for all* $\varepsilon > 0$;
(f) $\sum_{n=1}^{\infty} n^{-1} P\{\sup_{k \geq n} |(S_k - k\mu)/k^{1/t}| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$.

Proof. The proof that (a) $\Leftrightarrow$ (b) and (d) $\Leftrightarrow$ (e) is, except for details, the same as that given in Theorem 1 and will be omitted.

We proceed by showing that (a) $\wedge$ (b) $\Rightarrow$ (c) and (d) $\wedge$ (e) $\Rightarrow$ (f). Again we assume that $E(X_k) = 0$ if it exists. Since

$$P\left\{\sup_{k \geqq n} |S_k/k^{1/t}| > \varepsilon\right\} \leqq P\left\{\sup_{k \geqq 2^i} |S_k/k^{1/t}| > \varepsilon\right\}$$

for $2^i \leqq n < 2^{i+1}$ it is clearly sufficient to show that (b) or (e) $\Rightarrow$ $\sum_{i=1}^{\infty} P\{\sup_{k \geqq 2^i} |S_k/k^{1/t}| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. Now observe that

$$\sum_{n=1}^{\infty} n^{-1} \lg n P(|S_n^s| > n^{1/t}\varepsilon) \geqq c \sum_{i=1}^{\infty} \lg(2^i) P(|S_{2^i}^s| > 2^{(i+1)/t}\varepsilon). \tag{4}$$

This follows since

$$P(S_{2^i}^s + X_{2^i+1}^s + \cdots + X_n^s > 2^{(i+1)/t}\varepsilon) \geqq P(S_{2^i}^s > 2^{(i+1)/t}\varepsilon, (X_{2^i+1}^s + \cdots + X_n^s) \geqq 0)$$
$$\geqq \frac{1}{2} P(S_{2_i}^s > 2^{(i+1)/t}\varepsilon).$$

Next note that the finiteness of the-right hand side of (4), for all $\varepsilon > 0$, $\Rightarrow$ that $\sum_{i=1}^{\infty} P\{\sup_{k \geqq 2^i} |S_k^s/k^{1/t}| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. This follows from

$$\begin{aligned} \sum_{i=1}^{\infty} P\left\{\sup_{k \geqq 2^i} |S_k^s/k^{1/t}| > \varepsilon\right\} &\leqq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left\{\max_{2^j < l \leqq 2^{j+1}} |S_l^s/l^{1/t}| > \varepsilon\right\} \\ &\leqq 2 \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} P\{|S_{2^{j+1}}^s| > 2^{(i+2)/t} (\varepsilon/2^{2/t})\} \\ &= 2 \sum_{j=1}^{\infty} j P\{|S_{2^{j+1}}^s| > 2^{(j+2)/t}(\varepsilon/2^{2/t})\} \end{aligned} \tag{5}$$

where we have used P. Levy's inequality [8, p. 247] for the second inequality. From the symmetrization inequalities [8, p. 247] it follows that

$$\sum_{i=1}^{\infty} P\left\{\sup_{k \geqq 2^i} |S_k^s/k^{1/t}| > \varepsilon\right\} < \infty \text{ for all } \varepsilon > 0$$

$$\Rightarrow \sum_{i=1}^{\infty} P\left\{\sup_{k \geqq 2^i} |S_k/k^{1/t} - \text{med}(S_k/k^{1/t})| > \varepsilon\right\} < \infty \text{ for all } \varepsilon > 0;$$

further, (a) or (d)$\Rightarrow$that $\operatorname{med}(S_k/k^{1/t})\to 0$ as $k\to\infty$ by [8, p. 242]. Thus the finiteness of the right-hand side of (4), for all $\varepsilon>0$, $\Rightarrow \sum_{i=1}^{\infty} P\{\sup_{k\geqq 2^i}|S_k/k^{1/t}|>\varepsilon\}<\infty$ for all $\varepsilon>0$ and again by the symmetrization inequalities it follows that (b) or (e) guarantees the finiteness of the left side of (4) for all $\varepsilon>0$.

To complete the proof of the theorem, we now show that (c) $\Rightarrow$(a) and (f) $\Rightarrow$(d). Again let $EX_k=0$. Define $a_{n+k}=P(|X_1|^t>(n+k))$ and choose $\varepsilon=1/2$. Then

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left\{\sup_{k\geqq n} |S_k/k^{1/t}|>1/2\right\}\\
&\geqq \sum_{n=1}^{\infty} n^{-1} P\left\{\bigcup_{k=1}^{\infty} [|X_{n+k}|>(n+k)^{1/t}]\right\} \qquad (6)\\
&\geqq \sum_{n=1}^{\infty} n^{-1}\left\{\sum_{k=1}^{\infty} a_{n+k}-\sum_{k=1}^{\infty} a_{n+k}\left[\sum_{j>k} a_{n+j}\right]\right\}.
\end{aligned}
$$

By Theorem 1, $E|X_1|^t<\infty$ and hence $\lim_{n\to\infty}\sum_{j=1}^{\infty} a_{n+j}=0$. Thus

$$
\infty > \sum_{n=1}^{\infty} n^{-1}\left\{\sum_{k=1}^{\infty} a_{n+k}\left(1-\sum_{j>k} a_{n+j}\right)\right\} \geqq c\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{\infty} a_{n+k},
$$

with $c>0$. This, however, implies that $\sum_{n=1}^{\infty} a_n \lg n<\infty$ which is equivalent to $E|X_1|^t\lg^+|X_1|<\infty$. In case $t\geqq 1$ we have as before that $EX_k=0$ and the proof is complete.

THEOREM 3. *Let $t>1$, $r>1$ and $1/2<r/t\leqq 1$. The following three statements are equivalent:*

(a) $E|X_k|^t<\infty$ *and* $EX_k=\mu$;

(b) $\sum_{n=1}^{\infty} n^{r-2}P\{|S_n-n\mu|>n^{r/t}\varepsilon\}<\infty$ *for all* $\varepsilon>0$;

(c) $\sum_{n=1}^{\infty} n^{r-2}P\{\sup_{k\geqq n}|(S_k-k\mu)/k^{r/t}|>\varepsilon\}<\infty$ *for all* $\varepsilon>0$.

Let $t>0$, $r>1$ and $r/t>1$. The following three statements are equivalent:

(d) $\sum|X_k^t|<\infty$;

(e) $\sum_{n=1}^{\infty} n^{r-2}P\{|S_n|>n^{r/t}\varepsilon\}<\infty$ *for all* $\varepsilon>0$;

(f) $\sum_{n=1}^{\infty} n^{r-2}P\{\sup_{k\geqq n}|S_k/k^{r/t}|>\varepsilon\}<\infty$ *for all* $\varepsilon>0$.

Proof. The proof that (b) $\Rightarrow$(c) and (e) $\Rightarrow$(f) is essentially that given in Theorem 2; only the details are different. That (c)$\Rightarrow$(b) and (f)$\Rightarrow$(e) is trivial. Thus to prove this theorem we need to show that (a) is equivalent to (b) and (d) to (e). The proof that (a) $\Rightarrow$(b) and (d) $\Rightarrow$(e) has been given in [6]. This proof is basically the same as the proof of Theorem 1; namely a systematic use of the truncation method and Markov inequality.

It remains only to prove that (b)$\Rightarrow$(a) and (e)$\Rightarrow$(d). We need consider only the case $1<r<2$ since for $r\geqq 2$ this result has been proven in [6] using the methods of [3]. However, for $1<r<2$, the proof in Theorem 1 that (b)$\Rightarrow$(a) and (d)$\Rightarrow$(c) applies equally well here to give the desired result. We omit the details.

Next we state as a lemma a result on infinite series that we need but to which we could find no reference in the literature.

LEMMA. *Let $\{a_n\}$ be a nonincreasing sequence of non-negative numbers converging to zero. Let $t \geqq 0$, then if $\sum_{n=1}^{\infty} n^t a_n < \infty$ it follows that $n^{t+1}a_n \to 0$.*

In particular if in Theorem 3 we let $r = t > 1$ it follows that $EX_k = 0$ and $E|X_k|^t < \infty \Rightarrow n^{t-1}P\{|S_n/n| > \varepsilon\} \to 0$ for all $\varepsilon > 0$. Next we apply the lemma to obtain a corollary to Theorem 3.

COROLLARY. *Let $EX_k = \mu$ and $E|X_k|^t < \infty$ for some $t \geqq 1$. Then the series $\sum_{n=1}^{\infty}(-1)^n n^{t-1}P\{\sup_{k \geqq n}|(S_k - k\mu)/k| > \varepsilon\}$ is finite for all $\varepsilon > 0$.*

Proof. We can assume $t > 1$ since for $t = 1$ we have an alternating series of terms monotonically decreasing to 0. Let $a_n = P\{\sup_{k \geqq n}|(S_k - k\mu)/k| > \varepsilon\}$ for arbitrary $\varepsilon > 0$, and let $T_{n,k} = \{n^{t-1}a_n - (n+1)^{t-1}a_{n+1} + \cdots \pm (n+k)^{t-1}a_{n+k}\}$. We will show that given $\delta > 0$ there exists $n(\delta)$ such that for $n \geqq n(\delta)$ and all k, $|T_{n,k}| \leqq \delta$. Since $\{a_n\}$ is a nonincreasing sequence it is clear that $T_{n,k}$ is minimized if $a_n = a_{n+1}$, $a_{n+2} = a_{n+3}$, etc. Thus

$$(7) \qquad T_{n,k} \geqq \begin{cases} -(t-1)\displaystyle\sum_{l=0}^{(k-1)/2}(n+2l+1)^{t-2}a_{n+2l} & \text{if } k \text{ odd} \\ -(t-1)\displaystyle\sum_{l=0}^{(k/2)-1}(n+2l+1)^{t-2}a_{n+2l} & \text{if } k \text{ even.} \end{cases}$$

On the other hand $T_{n,k} = \{n^{t-1}a_n + [-(n+1)^{t-1}a_{n+1} + \cdots \pm (n+k)^{t-1}a_{n+k}]\}$ and is clearly maximized if $a_{n+1} = a_{n+2}$, $a_{n+3} = a_{n+4}$, etc. Therefore

$$(8) \qquad T_{n,k} \leqq \begin{cases} n^{t-1}a_n + (t-1)\displaystyle\sum_{l=1}^{k/2}(n+2l)^{t-2}a_{n+2l} & \text{if } k \text{ even,} \\ n^{t-1}a_n + (t-1)\displaystyle\sum_{l=1}^{(k-1)/2}(n+2l)^{t-2}a_{n+2l} & \text{if } k \text{ odd.} \end{cases}$$

By Theorem 3 the series on the right-hand side of (7) and (8) converge to zero as $n \to \infty$ and from the lemma it follows that $n^{t-1}a_n \to 0$. This completes the proof.

The next theorem determines necessary and sufficient conditions for convergence rates to zero for the sequences

$$\{P(|S_n/n| > \varepsilon) : n \geqq 1\} \quad \text{and} \quad \left\{P\left\{\sup_{k \geqq n}|S_k/k| > \varepsilon\right\} : n \geqq 1\right\}.$$

THEOREM 4. *Let $t \geqq 0$. The following two statements are equivalent:*

(a) *$n^{t+1}P\{|X_k| > n\} \to 0$ and $\int_{|x|<n} x\,dF(x) \to 0$;*

(b) *$n^t P\{|S_n| > n\varepsilon\} \to 0$ for all $\varepsilon > 0$.*

If $t > 0$, the above two statements are equivalent to:

(c) *$n^t P\{\sup_{k \geqq n}|S_k/k| > \varepsilon\} \to 0$ for all $\varepsilon > 0$.*

Proof. Note that the case $t=0$ is just a restatement of the Weak Law of Large Numbers for identically distributed random variables. In proving that (a) $\Rightarrow$ (b) it is convenient to make the proof for symmetrized random variables X_k^s which, by the symmetrization inequalities, also satisfy the hypothesis of (a). We will prove $n^t P\{|S_n^s| > n\varepsilon\} \to 0$ for all $\varepsilon > 0$. This implies $n^t P\{|S_n/n - \operatorname{med}(S_n/n)| > \varepsilon\} \to 0$ for all $\varepsilon > 0$. However, the hypotheses of (a) imply $S_n/n \to 0$ in probability and thus $\operatorname{med}(S_n/n) \to 0$, completing the proof.

Let $S_{nn}^s = \sum_{k=1}^n X_{kn}^s$ where X_{kn}^s denotes the kth, $k \leqq n$, symmetrized random variable truncated at n. Then

$$n^t P\{|S_n^s| > n\varepsilon\} \leqq n^{t+1} P\{|X_1^s| > n\varepsilon\} + n^t P\{|S_{nn}^s| > n\varepsilon\},$$

and since $n^{t+1} P\{|X_1^s| > n\varepsilon\} \to 0$ it remains to prove $n^t P\{|S_{nn}^s| > n\varepsilon\} \to 0$. Choose r to be an even integer greater than $2t+1$. From Markov's inequality we obtain

$$n^t P\{|S_{nn}^s| > n\varepsilon\} \leqq cn^{t-r} E(S_{nn}^s)^r \leqq cn^{t-r}\{nE(X_{1n}^s)^r + n(n-1)E(X_{1n}^s)^{r-2}E(X_{1n}^s)^2 + \cdots\}.$$

Let $\{2i_1, 2i_2, \cdots, 2i_m\}$ be a partition of r into positive even integers. A bound for the corresponding term in the preceding expansion is then given by $cn^{t-r+m}E(X_{1n}^s)^{2i_1} \cdots E(X_{1n}^s)^{2i_m}$. By hypothesis $n^{t+1}P\{|X_1^s| > n\} \to 0$ and thus upon integrating by parts we obtain that all factors $E(X_{1n}^s)^{2i_j}$ for which $2i_j < t+1$ are bounded, those with $2i_j = t+1$ are $o(\lg n)$ and those with $2i_j > t+1$ are $o(n^{2i_j - t - 1})$. Consequently $cn^{t-r+m}E(X_{1n}^s)^{2i_1} \cdots E(X_{1n}^s)^{2i_m}$ is bounded by a product $a_n b_n$ where $a_n = O(n^{-u})$ and $b_n = o(n^{t-[vt+wt]} \lg^v n)$ with $u = \sum_{i_j < (t+1)/2} (2i_j - 1)$, $v =$ number of $i_j = (t+1)/2$ and $w =$ number of $i_j > (t+1)/2$. It follows from this that for $r > 2t+1$ all partitions of r into even integers yield $o(1)$ terms. Thus we have shown that (a) $\Rightarrow$ (b).

Now we prove that (b) $\Rightarrow$ (a).

$$\begin{aligned}
P\{S_n^s > n\varepsilon\} &\geqq P \bigcup_{i=1}^n \left\{ (X_i^s > n\varepsilon) \cap \left(\sum_{j=1;\, j\neq i}^n X_j^s \geqq 0 \right) \right\} \\
&\geqq \sum_{i=1}^n \left\{ \frac{1}{2} P(X_i^s > n\varepsilon) - P(X_i^s > n\varepsilon) \sum_{j=1;\, j \neq i}^n P(X_j^s > n\varepsilon) \right\} \\
&= \sum_{i=1}^n P(X_i^s > n\varepsilon) \left\{ \frac{1}{2} - (n-1) P(X_i^s > n\varepsilon) \right\} \\
&\geqq nP(X_i^s > n\varepsilon) \left[\frac{1}{2} - \delta \right]
\end{aligned}$$

where δ can be chosen arbitrarily close to zero for sufficiently large n. Hence $n^t P\{|S_n| > n\varepsilon\} \to 0 \Rightarrow n^t P\{|S_n^s| > n\varepsilon\} \to 0 \Rightarrow n^{t+1} P\{|X_i^s| > n\varepsilon\} \to 0 \Rightarrow n^{t+1} P\{X_i| > n\varepsilon\} \to 0$. Finally the hypotheses of (b) imply that $S_n/n \to 0$ in probability and thus by the Weak Law of Large Numbers [8, p. 278] $\int_{|x|<n} x\, dF(x) \to 0$.

The proof is completed by showing that (b) $\Rightarrow$ (c) since clearly (c) $\Rightarrow$ (b). Let $2^{i-1} < n \leqq 2^i$, then it follows from Lévy's inequalities that

$$P\left\{\sup_{k\geqq n}\left|\frac{S_k^s}{k}\right|\geqq\varepsilon\right\}\leqq 2\sum_{j=i}^{\infty}P\left\{\left|\frac{S_{2^j}^s}{2^j}\right|\geqq\varepsilon/2\right\}.$$

Thus

$$n^tP\left\{\sup_{k\geqq n}\left|\frac{S_k^s}{k}\right|\geqq\varepsilon\right\}\leqq 2\sum_{j=i}^{\infty}\left[2^{jt}P\left\{\left|\frac{S_{2^j}^s}{2^j}\right|\geqq\varepsilon/2\right\}\right]2^{-(j-i)t}\leqq\left[\frac{2^t}{2^t-1}\right]2\delta$$

where i has been chosen so large that $n^tP\{|S_n^s/n|\geqq\varepsilon/2\}\leqq\delta$ for all $n\geqq 2^i$. Since (b) $\Rightarrow\operatorname{med}(S_k/k)\to 0$, we conclude that $n^tP\{\sup_{k\geqq n}|S_k/k|\geqq\varepsilon\}\to 0$.

In the preceding theorems we have obtained convergence rates for expressions of the form $P(|S_n|>n^\alpha\varepsilon)$ and $P(\sup_{k\geqq n}|S_k/k^\alpha|>\varepsilon)$ with $\alpha>1/2$ and $\varepsilon>0$. Now we consider the problem of convergence rates when ε is replaced by a sequence $\{e_n\}$ decreasing to zero. Thus we are led to consider convergence rates of sequences $\{P(S_n>b_n)\colon n\geqq 1\}$ when $\{b_n\}$ belongs to the upper class of $\{S_n\}$. In [5] Feller has given a criterion for a sequence to be in the upper class: Let $\{X_k\colon k\geqq 1\}$ be a sequence of independent and identically distributed random variables with $EX_k=0$, $EX_k^2=1$ and $EX_k^2(\lg^+|X_k|)^{1+\delta}<\infty$ for some $\delta>0$. Let $\phi(t)$ be a positive monotonically increasing function. Then $P(S_n>\sqrt{n}\phi(n)\text{ i.o.})=0$ if and only if $\int_1^\infty(\phi(t)/t)e^{-(1/2)\phi^2(t)}dt<\infty$.

For sequences in the upper class we have the following results on convergence rates.

THEOREM 5. *Let $\{X_k\colon k\geqq 1\}$ and $\phi(t)$ satisfy the hypotheses of Feller's criterion. Then*

$$\sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}P(S_n>\sqrt{n}\phi(n))$$

converges if and only if

$$\int_1^\infty\frac{\phi(t)}{t}e^{-(1/2)\phi^2(t)}dt<\infty.$$

Proof. Let $\Phi(x)$ denote the distribution function of a normal random variable with mean zero and variance 1 and let $F_n(x)=P(S_n/\sqrt{n}\leqq x)$. By hypothesis $EX_k=0$, $EX_k^2=1$ and $EX_k^2(\lg^+|X_k|)^{1+\delta}<\infty$ and hence by [7]

$$\sup_{x\in\mathcal{R}}|F_n(x)-\Phi(x)|\leqq C/(\lg n)^{1+\delta}.$$

Therefore by a result of Esseen [4, p. 70] it follows that

$$|F_n(x)-\Phi(x)|\leqq\frac{1}{1+x^2}\left\{\sqrt{\frac{2}{\pi}}\frac{4\lg\lg n+1}{2(\lg\lg n)^{1/2}}\frac{1}{(\lg n)^2}+\frac{C\lg\lg n}{(\lg n)^{1+\delta}}\right\}.$$

Now

$$\sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}P(S_n>\sqrt{n}\phi(n))$$

$$=\sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}[F_n(\phi(n))-\Phi(\phi(n))]+\sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}(1-\Phi(\phi(n)))$$

and setting $x = \phi(n)$ in the previous inequality the first term on the right-hand side of this equation is absolutely convergent. Therefore the left side of the equation converges and diverges with $\sum_{n=1}^{\infty}(\phi^2(n)/n)(1-\Phi(\phi(n)))$. However,

$$C\sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}\left(\frac{1}{\phi(n)}-\frac{1}{\phi^3(n)}\right)e^{-(1/2)\phi^2(n)} \leqq \sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}(1-\Phi(\phi(n)))$$
$$\leqq C\sum_{n=1}^{\infty}\frac{\phi(n)}{n}e^{-(1/2)\phi^2(n)}$$

and consequently from the integral convergence test it follows that

$$\sum_{n=1}^{\infty}\frac{\phi^2(n)}{n}(1-\Phi(\phi(n))) < \infty$$

if and only if $\int_1^\infty(\phi(t)/t)e^{-(1/2)\phi^2(t)}dt < \infty$, completing the proof.

If we let $\phi(n) = (1+\varepsilon)(2\lg\lg n)^{1/2}$, for $\varepsilon > 0$, we obtain

Theorem 6. *If $\{X_k : k \geqq 1\}$ satisfies the hypotheses of Feller's criterion then*

$$\sum_{n=3}^{\infty}(n\lg n)^{-1}P\left\{\sup_{k\geqq n}\frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon\right\} < \infty$$

for all $\varepsilon > 0$.

Proof. It is clearly sufficient to prove the result for all ε contained in the interval (0, 1). Choose and fix such an ε. Let γ denote a number in (1,2) such that $(1+\varepsilon/2)/\gamma > 1$. Using $[\alpha]$ to denote the largest integer contained in α, let i_0 denote the smallest positive integer such that (a) $[\gamma^i] < [\gamma^{i+1}]$ for all $i \geqq i_0 - 2$, (b) $\lg\lg[\gamma^{i_0-1}] > (2\gamma)^2/\varepsilon^2$, and (c) $\gamma^{i_0-2} > 1/(\gamma-1)$. Finally let n_0 be the largest integer such that $n_0 \leqq [\gamma^{i_0}]$ and for $n \geqq n_0$ define i_n to be the smallest integer such that $n < [\gamma^{i_n}]$. Now note that

$$\sum_{n=n_0}^{\infty}(n\lg n)^{-1}P\left\{\sup_{k\geqq n}\frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon\right\}$$
$$\leqq \sum_{n=n_0}^{\infty}(n\lg n)^{-1}\left\{P\left(\max_{n\leqq k<[\gamma^{i_n}]}\frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon\right)\right.$$
$$\left.+\sum_{i=i_n+1}^{\infty}P\left(\max_{[\gamma^i]\geqq k<[\gamma^{i+1}]}\frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon\right)\right\}.$$

Since $EX_k^2 < \infty$ it follows from Lévy's inequalities that

$$P\left(\max_{n\leqq k<[\gamma^{i_n}]}\frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon\right)$$
$$\leqq P\left(\max_{n\leqq k<[\gamma^{i_n}]} S_k > (1+\varepsilon)(2n\lg\lg n)^{1/2}\right)$$
$$\leqq 2P(S_{[\gamma^{i_n}]} \geqq (1+\varepsilon)(2n\lg\lg n)^{1/2} - (2[\gamma^{i_n}])^{1/2}).$$

From (b) and (c) it follows that $(1+\varepsilon)(2n\lg\lg n)^{1/2} - (2[\gamma^{i_n}])^{1/2} \geqq (1+\varepsilon/2)(2/\gamma^2 \lg\lg[\gamma^{i_n-1}])^{1/2}[\gamma^{i_n}]^{1/2}$ and thus we have that

$$P\left(\max_{n \leqq k < [\gamma^{i_n}]} \frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon \right)$$
$$\leqq 2P(S_{[\gamma^{i_n}]} \geqq (1+\varepsilon/2)(2/\gamma^2 \lg\lg[\gamma^{i_n-1}])^{1/2}[\gamma^{i_n}]^{1/2}).$$

In a similar manner we obtain that $P(\max_{[\gamma^{i_n}] \leqq k < [\gamma^{i_n+1}]} S_k/(2k\lg\lg k)^{1/2} > 1+\varepsilon) \leqq 2P(S_{[\gamma^{i_n+1}]} \geqq (1+\varepsilon/2)(2/\gamma^2\lg\lg[\gamma^{i_n}])^{1/2}[\gamma^{i_n+1}]^{1/2})$. Thus

$$\sum_{n=n_0}^{\infty} (n\lg n)^{-1} P\left\{ \sup_{k \geqq n} \frac{S_k}{(2k\lg\lg k)^{1/2}} > 1+\varepsilon \right\}$$
$$\leqq 2 \sum_{n=n_0}^{\infty} (n\lg n)^{-1} \sum_{i=i_n}^{\infty} P(S_{[\gamma^i]} \geqq (1+\varepsilon/2)(2/\gamma^2\lg\lg[\gamma^{i-1}])^{1/2}[\gamma^i]^{1/2}). \tag{9}$$

To prove that the right-hand side of (9) is finite it is sufficient to consider only the case when the X_k are normal random variables with mean zero and variance one. This follows since

$$\sup_{x \in \mathscr{R}} \left| P\left(\frac{S_{[\gamma^i]}}{[\gamma^i]^{1/2}} \geqq x \right) - \int_x^{\infty} 1/(2\pi)^{1/2} e^{-t^2/2} dt \right| \leqq C(\lg[\gamma^i])^{-1-\delta}$$

(by [7]) and $i_n \geqq \lg_\gamma n$. However, for centered normal random variables with variance one it follows from well known inequalities that the right side of (9) is bounded by $C\sum_{n=n_0}^{\infty}(n\lg n)^{-1}\sum_{i=i_n}^{\infty}(\lg[\gamma^{i-1}])^{-((1+\varepsilon/2)/\gamma)^2}$ and since $((1+\varepsilon/2)/\gamma)^2 > 1$ and $i_n \geqq \lg_\gamma n$ the above series is finite. This completes the proof.

Stationary random variables. Let $\{X_k : k \geqq 1\}$ be a stationary ergodic sequence with $EX_k = 0$. From the ergodic theorem it follows that $n^{-1}S_n \to 0$ a.e. and thus by analogy with Theorem 1 of this paper one might hope to show that $\sum_{n=1}^{\infty} n^{-1}P\{|S_n| > n\varepsilon\} < \infty$ for all $\varepsilon > 0$. However, the following construction demonstrates that this is not the case and in fact for the general ergodic stationary case no size restriction on the X_k can guarantee $\sum_{n=1}^{\infty} n^{-1}P(|S_n| > n\varepsilon) < \infty$ for all $\varepsilon >$

EXAMPLE. *There exists a stationary ergodic sequence such that* $|X_k| = 1$, $EX_k = 0$ *and* $\sum_{n=1}^{\infty} n^{-1}P\{|S_n| \geqq n\} = \infty$. We construct the example by defining a probability space $(\Omega, \mathscr{A}, P)$ an ergodic measure preserving transformation T on Ω and a function ϕ on Ω such that $|\phi| = 1$ and $\int \phi(\omega)P(d\omega) = 0$. The desired stationary sequence is $\{X_k = \phi(T^{k-1})\}$.

Let $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ where $\Omega_n = \{(x,n): 0 \leqq x < l_n\}$ with $\{l_n\}$ a nonincreasing sequence of non-negative numbers such that $\sum_{n=0}^{\infty} l_n = 1$. The l_n's are defined implicitly by

$$[l_n(\lg l_n)^2]^{-1} = n + C \qquad \text{for } n = 1, 2, \cdots,$$

$$l_0 = 1 - \sum_{n=1}^{\infty} l_n$$

with C chosen fixed and so large that $\sum_{n=1}^{\infty} l_n < 1/2$. This is possible since for large x one has

$$x(\lg x)^2/(\lg x + 2\lg\lg x)^2 < x < x(\lg x)^2 \lg\lg x/(\lg x + 2\lg\lg x + \lg\lg\lg x)^2$$

and consequently the solution, y, of $[y(\lg y)^2]^{-1} = x$ satisfies

$$[x(\lg x)^2 \lg\lg x]^{-1} < y < [x(\lg x)^2]^{-1}.$$

The class $\mathscr{A}$ of measurable sets is the class of unions of linear Lebesgue measurable sets and P is the sum of the linear Lebesgue measures. Let T_0 be a measure preserving ergodic transformation on Ω_0 and define T on Ω as follows:

$$T(x,y) = \begin{cases} (x, y+1) & \text{if } (x, y+1) \in \Omega, \\ (T_0 x, 0) & \text{if } (x, y+1) \notin \Omega. \end{cases}$$

T is an ergodic, measure preserving transformation on Ω. Let $A_0 \subset \Omega_0$ be a measurable subset of Ω_0 of measure $(1/2 - \sum_{n=1}^{\infty} l_n)$. Define ϕ on Ω as follows

$$\phi(\omega) = \begin{cases} +1, & \omega \in \left[\bigcup_{n=1}^{\infty} \Omega_n \cup A_0\right], \\ -1, & \omega \in \Omega_0 - A_0. \end{cases}$$

Let $X_k = \phi(T^{k-1})$, then clearly $\{X_k : k \geqq 1\}$ is a stationary, ergodic sequence such that $|X_k| = 1$ and $EX_k = 0$. For this process

$$(\omega : S_n(\omega) = n\} \supseteq \{(x,y) : (x, y+n-1) \in \Omega, y > 0\}$$

and hence

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P(S_n = n) &\geqq \sum_{n=1}^{\infty} (n+C)^{-1} \int_0^{l_n} \{[x(\lg x)^2]^{-1} - (n+C)\} dx \\ &= \sum_{n=1}^{\infty} (n+C)^{-1} \{-(\lg l_n)^{-1} - (n+C) l_n\} \qquad (10) \\ &\geqq \sum_{n=1}^{\infty} (n+C)^{-1} \{\lg [(n+C) \lg^2 (n+C) \lg\lg (n+C)]\}^{-1} \\ &\qquad - \sum_{n=1}^{\infty} l_n \\ &= \infty. \end{aligned}$$

In the converse direction let $t > 1$ and suppose $\{X_k : k \geqq 1\}$ is a stationary ergodic sequence, with $\sum_{n=1}^{\infty} n^{t-2} P(|S_n| > n\varepsilon) < \infty$ for all $\varepsilon > 0$. By the methods of [2, Theorem 1] it follows easily that $E|X_k|^{t-1} < \infty$. However, if in addition it is assumed that the random variables are independent then by Theorem 3 of this paper it follows that $E|X_k|^t < \infty$. Consequently it is of interest to investigate whether in the general stationary ergodic case one can prove that moments larger than the $(t-1)$st exist. The following example shows this is not the case.

EXAMPLE. *Let $t > 1$. There exists a stationary ergodic sequence $\{X_k: k \geqq 1\}$ such that $\sum_{n=1}^{\infty} n^{t-2}P(|S_n| > n\varepsilon) < \infty$ for all $\varepsilon > 0$, $E|X_k|^{t-1} < \infty$ and $E|X_k|^{t-1+\delta} = \infty$ for all $\delta > 0$.* We proceed as in the previous example. Define $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ with $\Omega_n = \{(x,n): 0 \leqq x < l_n\}$, l_n nonincreasing and $\sum_{n=0}^{\infty} l_n = 1$. $\mathscr{A}$ is the σ-field of Lebesgue measurable sets of Ω and P is Lebesgue measure on Ω. Let T_0 be a measure preserving, ergodic transformation on Ω_0 and T the extension as defined in the preceding example. Thus $(\Omega, \mathscr{A}, P)$ is a probability space with T an ergodic, measure preserving transformation on it. Finally we define the sequence $\{l_n: n \geqq 0\}$ and a function ϕ on Ω:

$$l_{2n} = l_{2n+1} = 2^{-n-2} \text{ for } n = 0,1,2,\cdots,$$

$$\phi(\omega) = \begin{cases} \left(\dfrac{2^n}{n^2}\right)^{1/(t-1)} & \text{if } \omega \in \Omega_{2n}, \quad n = 1,2,\cdots, \\ -\left(\dfrac{2^n}{n^2}\right)^{1/(t-1)} & \text{if } \omega \in \Omega_{2n+1}, \quad n = 1,2,\cdots, \\ 0 & \text{if } \omega \in \Omega_0 \cup \Omega_1. \end{cases}$$

Let $X_k = \phi(T^{k-1})$ for $k = 1,2,\cdots$. Then $E|X_k|^{t-1} = (1/2)\sum_{n=1}^{\infty} n^{-2} < \infty$ while for any $\delta > 0$ $E|X_k|^{t-1+\delta} = (1/2)\sum_{n=1}^{\infty} 2^{n\delta/(t-1)}n^{-[2+2\delta/(t-1)]} = \infty$. We finish by showing $\sum_{n=1}^{\infty} n^{t-2}P(|S_n| > n\varepsilon) < \infty$ for all $\varepsilon > 0$. This follows since at all points of Ω the summands X_k, $k = 1,\cdots,n$, occur in pairs of equal and opposite sign except possibly for X_1 and X_n. If both these are unpaired they are of opposite sign and different absolute value. Thus $|S_n| < n\varepsilon$ unless $\max(-X_1, X_n) \geqq n\varepsilon$. Therefore,

$$P\{|S_n| \geqq n\varepsilon\} \leqq P\{|\phi| \geqq n\varepsilon\} \leqq Cn^{1-t}(\lg_2 n)^{-2}$$

and hence

$$\sum_{n=1}^{\infty} n^{t-2}P\{|S_n| > n\varepsilon\} \leqq C \sum_{n=1}^{\infty} n^{-1}(\lg_2 n)^{-2} < \infty.$$

Independent, nonidentically distributed variables. In this section we will consider only sequences $\{X_k: k \geqq 1\}$ of independent, but not necessarily identically distributed, variables.

PROPOSITION 1. (a) $\sum_{n=1}^{\infty} n^{-1}P\{|S_n/n - \mathrm{med}(S_n/n)| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$ $\Rightarrow [S_n/n - \mathrm{med}(S_n/n)] \to 0$ *a.s. but not conversely.*

(b) *If in addition* $|X_i| < i$, *then* $\sum_{n=1}^{\infty} n^{-1}P\{|(S_n - ES_n)/n| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0 \Rightarrow (S_n - ES_n)/n \to 0$ *a.s.*

Proof. By the hypothesis of (a)

$$\infty > \sum_{n=1}^{\infty} n^{-1}P\{|S_n^s/n| > \varepsilon\} \geqq \frac{1}{4}\sum_{j=0}^{\infty} P\{|S_{2^j}^s/2^j| > 2\varepsilon\}$$

$$\geqq \frac{1}{8}\sum_{j=1}^{\infty} P\left\{\frac{|S_{2^j}^s - S_{2^{j-1}}^s|}{2^j} > 2\varepsilon\right\}.$$

Therefore, by the a.s. stability criterion [8, p. 252] we have $S_n^s/n \to 0$ a.s. and hence $[S_n/n - \mathrm{med}(S_n/n)] \to 0$ a.s. We prove that $[S_n/n - \mathrm{med}(S_n/n)] \to 0$ a.s. does not imply the finiteness of $\sum_{n=1}^{\infty} n^{-1}P\{|S_n/n - \mathrm{med}(S_n/n)| > \varepsilon\}$ for all $\varepsilon > 0$ by exhibiting a sequence of independent random variables for which this is the case. Let X_1 be a symmetric random variable such that $P\{X_1 > t\} = P\{X_1 < -t\} = (\log t)^{-1}$ for large t and define $X_2 = X_3 = \cdots \equiv 0$; then clearly $S_n/ \to 0$ a.s. but $\sum_{n=1}^{\infty} n^{-1}P\{|S_n/n| > 1\} = \infty$.

Under the hypothesis of (b) we may conclude, as in (a), that $S_n^s/n \to 0$ in probability and therefore that the characteristic function of S_n^s/n, say $g_n^s(u)$, converges to 1 uniformly in every finite interval. Further we have that $|X_i| < i$ and thus it follows from the truncation inequality [8, p. 196] that

$$2\sum_{i=1}^{n} \sigma^2(X_i/n) = \sum_{i=1}^{n} \sigma^2(X_i^s/n) \leqq -12\log g_n^s(1/2) \to 0.$$

Therefore $|\mathrm{med}(S_n/n) - E(S_n/n)| \leqq \{2\sigma^2(S_n/n)\}^{1/2} \to 0$ and an application of part (a) concludes the proof.

Now we present an example to show that without the size restrictions on the X_i's we cannot dispense with the $\mathrm{med}(S_n/n)$ terms. For $i \geqq 10$ let $u(i) = 2^{2^i}$. Define a sequence of independent variables as follows. $X_j \equiv 0$ except if $j = u(i)$ or $j = u(i) + 1$;

$$X_{u(i)} = \begin{cases} u(i) & \text{with probability } = 2^{u(i)}/[u(i) + 2^{u(i)}], \\ -2^{u(i)} & \text{with probability } = u(i)/[u(i) + 2^{u(i)}], \end{cases}$$

$$X_{u(i)+1} = \begin{cases} -u(i) & \text{with probability } = u(i)/[u(i) + 2^{u(i)}], \\ 2^{u(i)} & \text{with probability } = 2^{u(i)}/[u(i) + 2^{u(i)}]. \end{cases}$$

For this sequence $EX_j = 0$, $j = 1, 2 \cdots$. However, $\mathrm{med}(S_{u(i)}/u(i)) = 1$, and $\mathrm{med}(S_j/j) = 0$ for $j \neq u(i)$. Thus S_n/n does not converge to 0 a.s. but for each $\varepsilon > 0$ and each $t > 0$

$$\sum_{n=1}^{\infty} n^{-1}P\{|S_n/n| > \varepsilon\} \leqq 2\sum_{i=10}^{\infty} [u(i)/2^{u(i)}] \sum_{j=u(i)}^{2\cdot 2^{u(i)}/\varepsilon} j^{-t} + \sum_{i=10}^{\infty} [u(i)]^{-t} < \infty.$$

The classical Kolmogorov criterion for the Strong Law (and also Brunk's extended criteria) imply also the stronger form of convergence.

Proposition 2. *Let $\{X_k : k \geqq 1\}$ be a sequence of independent random variables with $E(X_k) = 0$ and $\sum_{k=1}^{\infty} E|X_k|^{2r}/k^{r+1} < \infty$ for some $r > 1$. Then for all $\varepsilon > 0$ $\sum_{n=1}^{\infty} n^{-1}P\{|S_n|/n > \varepsilon\} < \infty$.*

Proof.

$$\sum_{n=1}^{\infty} n^{-1}P\{|S_n| > n\varepsilon\} \leqq \sum_{n=1}^{\infty} n^{-1}Cn^{r-1}\left(\sum_{k=1}^{n} E|X_k|^{2r}\right)\frac{1}{n^{2r}}$$
$$\leqq C \sum_{k=1}^{\infty} \frac{E|X_k|^{2r}}{k^{r+1}} < \infty,$$

where we have used [8, p. 263, problems 4, 5] for the first inequality.

The following example demonstrates that Kolmogoroff's criterion $\sum \sigma_k^2/k^2 < \infty$ does not imply the stronger rate of convergence

$$\sum n^{-1}P\left\{\sup_{k\geqq n}\left|\frac{S_k}{k} - \operatorname{med}\frac{S_k}{k}\right| > \varepsilon\right\} < \infty$$

and a fortiori by Proposition 3 below it does not imply

$$\sum_{n=1}^{\infty} n^t P\{|S_n/n - \operatorname{med}(S_n/n)| > \varepsilon\} < \infty$$

for any $t > -1$.

Let X_k be symmetric independent variables with

$$\left.\begin{aligned} |X_k| &= k \quad \text{with probability} \quad \frac{1}{k(\lg k)^2}, \\ &= 0 \quad \text{with probability} \quad 1 - \frac{1}{k(\lg k)^2}, \end{aligned}\right\} \quad k = 3, 4, \cdots.$$

Then

$$\sum_{k=3}^{\infty} \sigma_k^2/k^2 = \sum_{k=3}^{\infty} \frac{1}{k(\lg k)^2} < \infty$$

while

$$\begin{aligned} \sum_{n=3}^{\infty} n^{-1}P\left\{\sup_{k\geqq n}\frac{|S_k|}{k} \geqq 1/2\right\} &\geqq \sum_{n=3}^{\infty} n^{-1}P\{|X_{n+k}| \geqq n+k \text{ for some } k \geqq 1\} \\ &= \sum_{n=3}^{\infty} n^{-1}\left\{1 - \prod_{k=n+1}^{\infty}\left(1 - \frac{1}{k(\lg k)^2}\right)\right\} \\ &\geqq \sum_{n=3}^{\infty} n^{-1} \sum_{k=n+1}^{\infty} \frac{1}{k(\lg k)^2}\left(1 - \sum_{j=n+1}^{\infty}\frac{1}{j(\lg j)^2}\right) \\ &\geqq C \sum_{n=3}^{\infty} \frac{1}{n \lg n} = \infty. \end{aligned}$$

PROPOSITION 3. *Let $\{X_k\}$ be a sequence of independent variables. If $t > -1$ then*

(a) $\sum_{n=1}^{\infty} n_t P\{|S_n/n - \text{med}(S_n/n)| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$ *if and only if* $\sum_{n=1}^{\infty} n^t P\{\sup_{k \geq n} |S_k/k - \text{med}(S_k/k)| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$.

If moreover $t \geqq 0$ *then also*

(b) $\sum_{n=1}^{\infty} n^t P\{|S_n/n| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$ *if and only if*

$$\sum_{n=1}^{\infty} n^t P\left\{\sup_{k \geq n} |S_k|/k > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$.

Proof. (a) follows by the methods of the proof of Theorem 2. If $t \geqq 0$

$$\sum_{n=1}^{\infty} n^t P\left\{\frac{|S_n|}{n} > \varepsilon\right\} < \infty \text{ for all } \varepsilon > 0 \Rightarrow \text{med}(S_n/n) \to 0$$

so (b) follows.

In the converse direction the methods of [2, p. 189] suffice to prove.

PROPOSITION 4. *For sequences $\{X_k : k \geqq 1\}$ of independent variables if*

$$\sum_{n=1}^{\infty} n^{-1} P\{|S_n| > n\varepsilon\} < \infty \text{ for all } \varepsilon > 0$$

then $E \lg^+ |X_k| < \infty$ *for all* k. *For* $t > 1$, *if*

$$\sum_{n=1}^{\infty} n^{t-2} P\{|S_n| > n\varepsilon\} < \infty \text{ for all } \varepsilon > 0$$

then $E|X_k|^{t-1} < \infty$ *for all* k.

That this result cannot be improved follows trivially by considering sequences for which $X_k = 0$, $k = 2, 3, \cdots$ and $E|X_1|^{t-1} < \infty$ but $E|X_1|^{t-1+\delta} = \infty$ for all $\delta > 0$.

In [2, p. 190] we obtained necessary and sufficient size restrictions on the individual independent variables X_k for an "exponential rate of convergence" of S_n/n. We have been unable to obtain such satisfactory necessary and sufficient conditions for the present rates of convergence: $\sum_{n=1}^{\infty} n^t P\{|S_n|/n > \varepsilon\} < \infty$. Proposition 1 of this section suggests that such a theorem for the case $t = -1$ is related to the classical problem of necessary and sufficient conditions for convergence with probability 1 of S_n/n.

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